The Goldberg-Seymour Conjecture on the edge coloring of multigraphs

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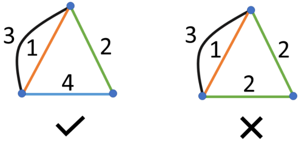
Joint work with Guantao Chen and Wenan Zang

Some notations

- Graphs in this talk may contain multiple edges but no loops.
- A (proper) k-edge-coloring φ of G is a mapping φ from $E(G)$ to $\{1, 2, \cdots, k\}$ (whose elements are called colors) such that no two incident edges receive the same color.



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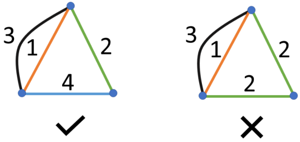

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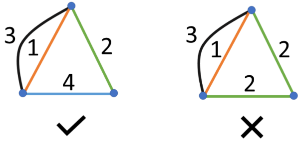

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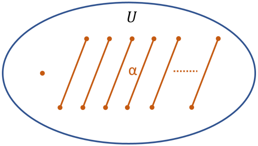
Another Lower Bound - Density

Let φ be a k-edge-coloring of G, U be an odd subset of $V(G)$ and E_α be the set of edges colored by α with both ends in U.

- $|E_\alpha| \leq \frac{|U|-1}{2}$.



- $|E(U)| = \sum_{\alpha \in [1,k]} |E_\alpha| \leq k \cdot \frac{|U|-1}{2}$, hence
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- $k \geq \omega := \max\left\{\frac{2|E(U)|}{|U|-1} : \ U \subseteq V, \ |U| \geq 3 \ and \ odd\right\}$.
- $\omega := \omega(G)$ is called the density of G.
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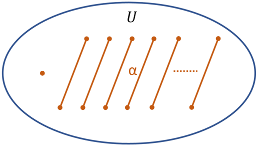

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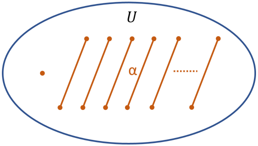

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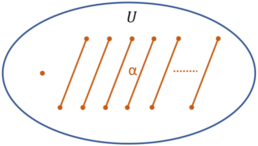

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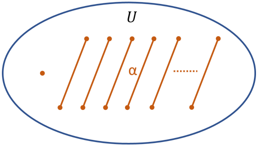

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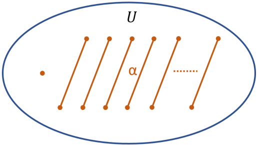

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Vizing's Theorem and the Goldberg-Seymour Conjecture

Shannon's bound (1949)

Let G be a multigraph. Then $\chi'(G) \leq \frac{3}{2}\Delta(G)$.

Vizing's Theorem (1964)

Let G be a multigraph with multiplicity μ. Then $\chi'(G) \leq \Delta(G) + \mu$.

The Goldberg-Seymour conjecture

In the 1970s, Goldberg and Seymour independently conjectured that $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \omega(G) \rceil\}$, which is equivalent to saying that if $\chi'(G) \geq \Delta + 2$, then $\chi'(G) = \lceil \omega(G) \rceil$.

The fractional chromatic index

Holyer in 1980 proved that determining the chromatic index of a graph is NP-complete, even when restricted to a simple cubic graph.

A fractional edge coloring of G is a non-negative weighting $w(.)$ of the set $\mathcal{M}(G)$ of matchings in G such that, for every edge $e \in E(G)$, $\sum_{M \in \mathcal{M}: e \in M} w(M) = 1$. Clearly, such a weighting $w(.)$ exists. The fractional chromatic index $\chi'_f := \chi'_f(G)$ is the minimum total weight $\sum_{M \in \mathcal{M}} w(M)$ over all fractional edge colorings of G. By definition, we have $\chi' \geq \chi'_f \geq \Delta$.

Consequences of the Goldberg-Seymour conjecture

Seymour showed that χ'_f can be computed in polynomial time and $\chi'_f(G) = \max\{\Delta(G), \omega(G)\}$.

So the Goldberg-Seymour conjecture implies that:

- A polynomial time algorithm to approximate the chromatic index within one color by computing χ'_f. In fact, determining the chromatic index is considered to be one of the easiest NP-complete problems in this sense.
- There are only two possible choices for $\chi'(G)$: $\max\{\Delta(G), \lceil\omega(G)\rceil\}$ and $\max\{\Delta(G)+1, \lceil\omega(G)\rceil\}$.

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Several related conjectures

Jakobsen (1973)

Let G be a critical graph and $\chi'(G) > \frac{m}{m-3}\Delta(G) + \frac{m-3}{m-1}$ for an odd integer $m \geq 3$, then $|V(G)| \leq m - 2$.

Seymour (1979)

If G is an r-regular graph such that $|\partial_G(X)| \geq r$ for every set $X \subseteq V(G)$ with $|X|$ odd (such a graph is said to be an r-graph), then G satisfies $\chi'(G) \leq r + 1$.

- $2|E(X)| \leq r|X| - r$ for every odd subset X.
- $\omega(G) = \frac{2|E(X)|}{|X|-1} \leq \frac{r|X|-r}{|X|-1} = r$.
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Kempe change

- A graph G is called critical if $\chi'(H) < \chi'(G)$ for any proper subgraph $H \subseteq G$. A graph G is called k-critical if it is critical and $\chi'(G) = k + 1$.

  For the rest of this talk, we let $G = (E, V)$ be a k-critical graph, $e \in E_G(x, y)$ be an edge of G and φ be a k-edge-coloring of $G - e$.
- For two colors α and β, an (α, β)-chain is a connected component of G induced by edges colored by α and β. An (α, β)-chain is called an (α, β)-path if it is indeed a path.
- Let P be an (α, β)-path (chain) under the k-edge coloring φ. Then φ' obtained from φ by interchanging colors α and β along P is also a k-edge coloring. This operation is called a Kempe Change, and is denoted by $\varphi' = \varphi/P$.

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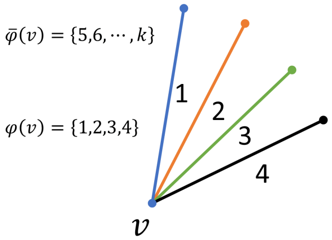
More notations

- For any $v \in V$, let $\varphi(v) := \{\varphi(e) : e \in E(v)\}$ denote the set of colors presented at v and $\overline{\varphi}(v) = \{1, 2, \cdots, k\} \backslash \varphi(v)$ the set of colors missing at v.



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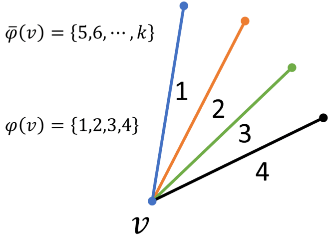

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An equivalent argument to the Goldberg-Seymour conjecture

The Goldberg-Seymour conjecture holds if and only if there is a vertex set $Z \subseteq V(G)$ with $e \in E(Z)$ which is both elementary and strongly closed for every k-critical graph G with $k > \Delta + 1$.

- A vertex set $X \subseteq V(G)$ is called elementary if $\overline{\varphi}(v) \cap \overline{\varphi}(w) = \emptyset$ for any two distinct vertices $v, w \in X$.
- An edge f is called a boundary edge of X if f has exact one end-vertex in X and denote by $\partial(X)$ the set of all boundary edges of X. We call X closed if there is no missing color in vertices of X are assigned to any edges in $\partial(X)$.
- A color α is called a defective color of X if it appears more than once on edges in $\partial(X)$. Moreover, a closed vertex set X is called strongly closed if there is no defective colors of X.

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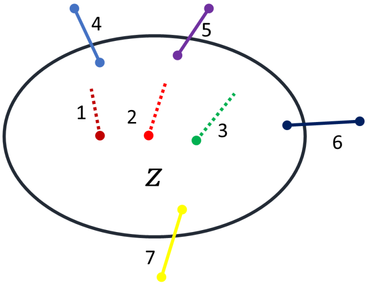

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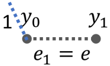
Too see this, let $Z \subseteq V(G)$ be a strongly closed elementary set. Then

- $|Z|$ is odd.
- Each color in $\overline{\varphi}(Z)$ induces $\frac{|Z|-1}{2}$ many edges in $G[Z]$.
- Each color in $\{1, 2, ..., k\} - \overline{\varphi}(Z)$ induces $\frac{|Z|-1}{2}$ many edges in $G[Z]$.
- This gives us $k\frac{|Z|-1}{2}$ many edges. With the uncolored edge e, we see that $\lceil \omega(Z) \rceil \geq k + 1 = \chi'(G)$, as desired.

Tashkinov Trees

A Tashkinov tree $T = (y_0, e_1, y_1, e_2, \cdots, y_{p-1}, e_p, y_p)$ is an alternating sequence of distinct vertices y_i and edges e_i of G, such that the endvertices of each e_i are y_{i+1} and y_r for some $r \in \{1, 2, \ldots, i\}$, $e_1 = e$ and $\varphi(e_i)$ is missing at y_j for some $j < i$. Note that the edge set of T indeed forms a tree.



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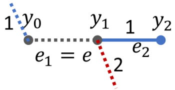

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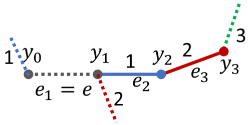

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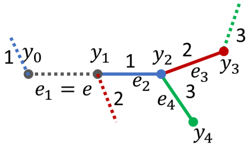

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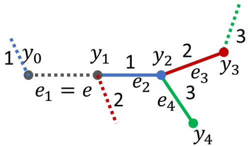
A Tashkinov tree $T = (y_0, e_1, y_1, e_2, \cdots, y_{p-1}, e_p, y_p)$ is an alternating sequence of distinct vertices y_i and edges e_i of G, such that the endvertices of each e_i are y_{i+1} and y_r for some $r \in \{1, 2, \ldots, i\}$, $e_1 = e$ and $\varphi(e_i)$ is missing at y_j for some $j < i$. Note that the edge set of T indeed forms a tree.



Weakness of the Tashkinov trees.

- A Tashkinov tree may not be strongly closed, though it could be closed.
- A Tashkinov tree must have each edge added with a color missing at a vertex before that edge.

Generalization

- A tree sequence $T = (y_0, e_1, y_1, e_2, \cdots, y_{p-1}, e_p, y_p)$ is an alternating sequence of distinct vertices y_i and edges e_i of G, such that the endvertices of each e_i are y_{i+1} and y_r for some $r \in \{1, 2, \ldots, i\}$.
- If a tree sequence T is not closed, the algorithm of adding an edge $f \in \partial(T)$ and the corresponded vertex with $\varphi(f) \in \overline{\varphi}(T)$ to T is called Tashkinov Augmenting Algorithm (TAA).
- A closure $\overline{T}$ of T is a tree-sequence obtained from T by applying TAA repeatedly until T is closed.

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New ideas

- For any color set C, an edge-k-coloring φ^* of $G - e$ is (T, C, φ)-stable if the following two properties hold.
  1. $\varphi^*(f) = \varphi(f)$ for every edge f incident to T with $\varphi(f) \in \overline{\varphi}(T) \cup C$.
  2. $\overline{\varphi}^*(v) = \overline{\varphi}(v)$ for any $v \in V(T)$, which gives $\overline{\varphi}^*(T) = \overline{\varphi}(T)$.

  We say a coloring φ^* is $(\emptyset, \emptyset, \varphi)$-stable if φ^* is an edge-k-coloring φ^* of $G - e$.

Goals of this concept

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- Colors α and β are T-interchangeable if there is at most one (α, β)-path intersecting T.

Goal of this concept

- To have colors interchanging with defective colors along a color alternating chain.

Example

Let T be a closed Tashkinov tree of G under φ. Then every color $\alpha \in \overline{\varphi}(T)$ is interchangeable with every color in φ for T.

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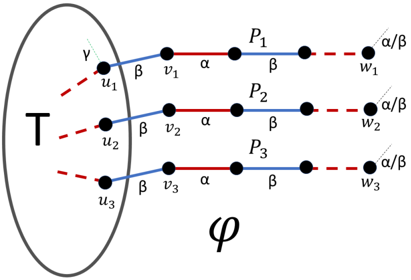
A proof of the example



Consider a counter-example as the above figure where α and β are not interchangeable, and assume among all the counter-examples, $|P_1| + |P_2| + |P_3| = L$ is minimum.

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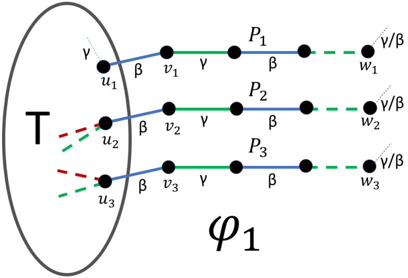

Note that T is still a Tashkinov tree under φ_1 obtained from φ by switching γ and α outside of T.

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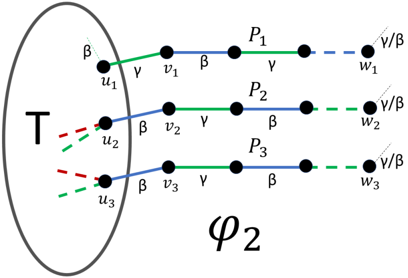
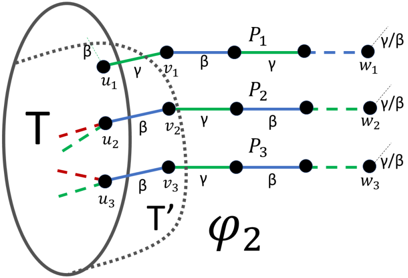


Let $\varphi_2 = \varphi_1 / P_1$.

A proof of the example



Let T' be a closed Tashkinov tree obtained from $T(u_1)$ after adding the vertices v_2 and v_3. Note that we have a contradiction to the elementariness of the Tashkinov trees if one of w_1, w_2, w_3 is contained in T', and a contradiction to $|P_1| + |P_2| + |P_3| = L$ being minimum otherwise.

Ideas

- Starting from an elementary tree sequence T which is not closed, we reserve two interchangeable colors for each defective color before it is missing when applying TAA to find a closure of T through a few "steps", and prove its closure is elementary.
- Starting from an closed elementary tree sequence T, we find a new vertex $v \notin T$ such that $T \cup v$ is elementary if T is not strongly closed.

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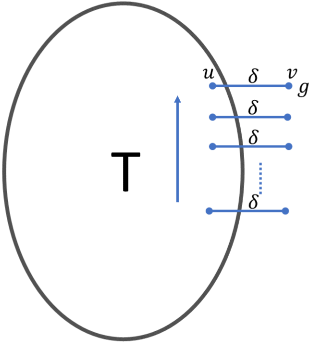
For the second point, we need to introduce two main types of extensions if T is closed but not strongly closed. Assume δ is a defective color in this case.



- Series Extension (SE): If $T \cup \{g, v\}$ is elementary under all (T, C, φ)-stable colorings, we extend $T \cup \{g, v\}$ to its closure under φ.

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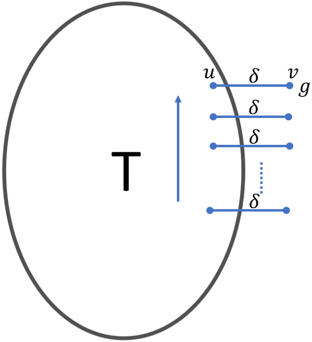

- Parallel Extension (PE): If the SE requirement is not satisfied, we can then assume that there exists a color $\gamma \in \overline{\varphi}(u) \cup \overline{\varphi}(v)$. Recolor the edge g by γ, update the coloring φ, and find a closure of T under this new coloring.

Elementariness of PE extensions

For a PE extension, we have a different situation than an SE extension, as we only add one more vertex to an SE extension to start this phase of our induction. Let $T(u)$ be the subsequence of T ending at u. Then a closure $T' := \overline{T(u)}$ contains at least two vertices not in T. We claim that $T \cup T'$ is elementary under the updated φ.

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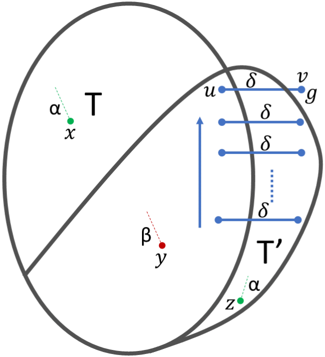

Main Theorem

Let n be a nonnegative integer and (G, e, φ) be a k-triple with $k \geq \Delta + 1$. Then for every ETT T satisfying MP with ladder $T_0 \subset T_1 \subset \cdots \subset T_n \subset T$ and coloring sequence $(\varphi_0, \varphi_1, \ldots, \varphi_n)$, the following six statements hold.

A1: For any positive integer l with $l \leq n$, if v_l is a supporting vertex and $m(v_l) = j$, then every (T_l, D_l, φ_l)-stable coloring φ_l^* is $(T_{v_l} - \{v_l\}, D_{j-1}, \varphi_{j-1})$-stable, particularly, φ_l^* is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable. For any two supporting vertices v_s and v_t with $s, t \leq n$, if $m(v_s) = m(v_t)$ but $v_s \neq v_t$, then $S_s \cap S_t = \emptyset$.

A2: If $\Theta_n = PE$, then under any (T_n, D_n, φ_n)-stable coloring φ_n^*, we have $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n = \{v_n\}$ where $S_n = \{\delta_n, \gamma_n\}$.

Main Theorem

A3: For any (T_n, D_n, φ_n)-stable coloring φ_n^*, if δ is a defective color of T_n under φ_n^* and $v \in a(\partial_{\varphi_n^*,\delta}(T_n))$ where v is not the smallest vertex along $\prec_\ell$ in $a(\partial_{\varphi_n^*,\delta}(T_n))$, then $v \prec_\ell v_i$ for any supporting or extension vertex v_i with $i \geq m(v)$.

A4: Every (T_n, D_n, φ_n)-stable coloring φ_n^* is a $\varphi_n \mod T$ coloring and every corresponding ETT T^* obtained from T_n under φ_n^* using the same extension type as $T_n \to T$ also satisfies MP.

A5: T is elementary under φ_n.

A6: T has Interchangeable Missing Colors property if T is closed.

Thanks for your attention!